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The Interval Symmetric Single-Step *ISS1* Procedure for Simultaneously Bounding Simple Polynomial Zeros

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ABSTRACT

The interval single-step procedure *IS1* established by Alefeld and Herzberger (1983) has been modified. The idea of Aitken (1950) and Alefeld (1977) is used to establish the interval symmetric single-step procedure *ISS1*.This procedure has a faster convergence rate than does *IS1*. In this paper, the convergence analysis of the procedure ISS1 *using* interval arithmetic (Moore (1962, 1979), Alefeld and Herzberger (1983)) is shown. The procedure *ISS1* is considered as the interval version of the point symmetric single-step procedure *PSS1* Monsi (2010).

Keywords: Interval analysis, interval procedure, simultaneous inclusion, simple zeros, *R*-order of convergence, *R*-factor of a sequence.

INTRODUCTION

Several interval iterative procedures for the simultaneous inclusion of simple polynomial zeros exist. See, for examples, Gargantini (1975, 1976, 1978, 1981), Garganti and Henrici (1972), Glatz (1975), Henrici (1974), Krier and Spellucci (1975), Milovanovic and Petkovic (1983), Petkovic (1980, 1982), Petkovic and Milovanovic (1983), Petkovic and Stefanovic (1986, 1987). Interval iterative procedures for simultaneous inclusion of simple polynomial zeros determine bounded closed intervals each of which contains an exact polynomial zero. Furthermore the widths of intervals are limited only by the precision of the machine floating point arithmetic. Thus interval iterative procedures can be used to determine very narrow computationally rigorous bounds on polynomial zeros.

The purpose of this paper is to describe the interval symmetric single-step procedure *ISS1* for simultaneously bounding simple polynomial zeros. The procedure *ISS1* is the interval version of the point symmetric single-step procedure *PSS1* Monsi (2010). The significance of using interval analysis (Moore (1962, 1979), Alfeled and Herzbeger (1983)) for determining the convergence rate of the procedure *ISS1* is that its convergence analysis is very straight forward.

The *R*-order of convergence analysis of an iterative procedure is used in this paper as a measure of the asymptotic convergence rate of the procedure. The concept of *R*-order of convergence is discussed in detail in Ortega and Rheinboldt (1970) and Alefeld and Herzberger (1983). The *R*-order of the procedure *I* which converges to x^* is denoted by $O_R(I,x^*)$ and the *R*-factor of a null sequence $w^{(k)}$ generated from the procedure *I* is denoted by $R_p(w^{(k)})$, where $p \ge 1$ and $w^{(k)}$ is a null sequence generated from the procedure *I*.

Furthermore, if there exists a $p \ge 1$ such that for any null sequence $\{w^{(k)}\}$ generated from $\{x^{(k)}\}$, then the *R*-factor of such sequence is defined to be

$$R_p(w^{(k)}) = \begin{cases} \lim_{k\to\infty} \sup \left\| w^{(k)} \right\|^{1/k}, & p = 1 \\ \\ \lim_{k\to\infty} \sup \left\| w^{(k)} \right\|^{1/p^k}, & p > 1, \end{cases}$$

where R_p is independent of the norm $\|\cdot\|$.

We may now define the *R*-order of the iteration *I* as

$$O_R(I,x^*) = \begin{cases} +\infty \ \mathit{if}\ R_p(I,x^*) = 0 & \text{for } \ p \ge 1 \\ \inf\left\{ p \,\middle|\, p \in [1,\infty), R_p(I,x^*) = 1 \right\} & \text{otherwise.} \end{cases}$$

Suppose that $R_p(w^{(k)}) < 1$ then it follows from Ortega and Rheinboldt (1970) that the *R*-order of I satisfies the inequality $O_R(I,x^*) \ge p.$ We will use this result in order to calculate the *R*-order of convergence of *ISS1* in the subsequent section.

The proof of the following theorem is in Ortega and Rheinboldt (1970).

Theorem 1

Let I be an iteration procedure with the limit x^*, and let $\Omega(I, x^*)$ be the set of all sequences $\{x^{(k)}\}$ generated by I having the properties that $\lim_{k\to\infty} x^{(k)} = x^*$ and $x^* \subseteq x^{(k)}, k \geq 0$. If there exists a $p \geq 1$ and a constant γ such that for all $\left\{x^{(k)}\right\} \in \Omega(I, x^*)$ and for a norm $\|\cdot\|$, it holds that $\left\|h^{(k+1)}\right\| \geq \gamma \left\|h^{(k+1)}\right\|^p, k \geq k(\{x^{(k)}\})$, then it follows that the R-order of I satisfies the inequality $O_R(I, x^*) \geq p$. □

THE INTERVAL TOTAL-STEP AND SINGLE-STEP PROCEDURES

Let $p: R^1 \rightarrow R^1$ be a polynomial of degree n defined by

$$p(x) = \sum_{i=0}^{n} a_i x^i \tag{1}$$

where $a_i \in R^1$ $(i = 0, \dots, n)$ are given. Suppose that p has n distinct zeros $x_i^* \in R$ $(i = 1, \dots, n)$ and that $\underline{x}_i^{(0)} \in I(R)$ $(i = 1, \dots, n)$ are such that

$$x_i^* \in \underline{x}_i^{(0)} (i = 1, \dots, n), \tag{2}$$

and

$$\underline{x}_i^{(0)} \cap \underline{x}_j^{(0)} \quad (i, j = 1, \dots, n;\ i \neq j). \tag{3}$$

It is assumed henceforth that $a_n = 1$, so that

$$p(x) = \prod_{j=1}^{n} \left(x - x_j^*\right). \tag{4}$$

By (4), for $i=1,\ldots,n\left(\forall x \neq x_j^* \left(j=1,\ldots,n\right)\right)$

$$x_j^* = x - \frac{p(x)}{\Pi_{j\neq i}\left(x - x_j^*\right)}. \tag{5}$$

If

$$x_i^{(0)} = m\left(\underline{x}_i^{(0)}\right)(i=1,\ldots,n), \tag{6}$$

are the midpoints of the intervals $\underline{x}_i^{(0)}(i=1,\ldots,n)$ respectively. Then by (2), (3)

$$x_i^{(0)} \neq x_j^* \quad (i,j=1,\ldots,n;\ \ j\neq i). \tag{7}$$

So by (5),

$$x_j^* = x_i^{(0)} - \frac{p(x_i^{(0)})}{\Pi_{j\neq i}\left(x_i^{(0)} - x_j^*\right)} \quad (i=1,\ldots,n). \tag{8}$$

Furthermore, by (3), (6), $x_i^{(0)} \notin \underline{x}_j^{(0)}(i,j=1,\ldots,n; j\neq i)$ whence

$$0 \notin \prod_{j\neq i}\left(x_i^{(0)} - \underline{x}_j^{(0)}\right) \quad (i=1,\ldots,n). \tag{10}$$

So by (2), (8), and the inclusion monotonicity (Alfeld and Herzberger (1983)) of real interval arithmetic,

$$x_i^* \in \underline{x}_i^{(1)} = \left\{ x_i^{(0)} - \frac{p\left(x_i^{(0)}\right)}{\Pi_{j\neq i}\left(x_i^{(0)} - \underline{x}_j^{(0)}\right)} \right\} \cap \underline{x}_i^{(0)} \ \ (i=1,\ldots,n). \tag{11}$$

This gives rise to the total-step procedure *IT1* of Alefeld and Herzberger (1983) defined by

$$x_i^{(k)} = m\left(\underline{x}_i^{(k)}\right) \ (i = 1,\dots,n), \tag{12a}$$

$$\underline{x}_i^{(k+1)} = \left\{ x_i^{(k)} - \frac{p\left(x_i^{(k)}\right)}{\Pi_{j\neq i}\left(x_i^{(k)} - \underline{x}_j^{(k)}\right)} \right\} \cap \underline{x}_i^{(k)} \ (i = 1,\dots,n)(k \geq 0), \tag{12b}$$

which may be regarded as an interval version of the procedure *PT1* in Monsi (2010). The following theorems are proved in Alefeld and Herzberger (1983).

Theorem 2

If (i) (2) and (3) hold; (ii) the sequences $\left\{\underline{x}_j^{(k)}\right\}\left(i = 1,\dots,n\right)$ are generated from (12), then $\left(\forall k \geq 0\right) x_i^* \in \underline{x}_i^{(k+1)} \subseteq \underline{x}_i^{(k)} \left(i = 1,\dots,n\right)$. If also (iii) $0 \notin \underline{d}_i$ where $\underline{d}_i = \left[d_{iI}, d_{iS}\right] \in I(R)$ is such that $p'(x) \in \underline{d}_i \left(\forall x \in \underline{x}_i^{(0)}\right)\left(i = 1,\dots,n\right)$, then $\underline{x}_i^{(k)} \to x_i^* \left(k \to \infty\right)\left(i = 1,\dots,n\right)$ and $\left(\forall k \geq 0\right)\left(i = 1,\dots,n\right)$

$$w\left(\underline{x}_i^{(k+1)}\right) \leq \frac{1}{2}\left(1 - \frac{d_{iI}}{d_{iS}}\right) w\left(\underline{x}_i^{(k)}\right), \tag{13}$$

where $w\left(\underline{x}_i^{(k)}\right) = w\left(\left[x_{iI}^{(k)}, x_{iS}^{(k)}\right]\right) = x_{iS}^{(k)} - x_{iI}^{(k)}$. Furthermore, for $i = 1,\dots,n$, $O_R\left(IT1, x_i^*\right) \geq 2$. □

The interval single-step procedure *IS1* of Alefeld and Herzberger (1983) is the interval version of the point single-step procedure *PS1* which is discussed in Monsi (2010), and consists of generating the sequences $\left\{\underline{x}_i^{(k)}\right\}\left(i = 1,\dots,n\right)$ from

$$x_i^{(k)} = m\left(\underline{x}_i^{(k)}\right)\left(i = 1,\dots,n\right) \tag{14a}$$

$$\underline{x}_i^{(k+1)} = \left\{ x_i^{(k)} - \frac{p\left(x_i^{(k)}\right)}{\Pi_{j=1}^{i-1}\left(x_i^{(k)} - \underline{x}_j^{(k+1)}\right)\Pi_{j=i+1}^{n}\left(x_i^{(k)} - \underline{x}_j^{(k)}\right)} \right\} \cap \underline{x}_i^{(k)} \qquad (14b)$$

$$\left(i = 1,...,n\right)\left(k \geq 0\right).$$

Theorem 3

If (i) (2) and (3) hold; (ii) the sequences $\left\{\underline{x}_i^k\right\}\left(i = 1,...,n\right)$ are generated from (14), then $\left(\forall k \geq 0\right) x_i^* \in \underline{x}_i^{(k+1)} \subseteq \underline{x}_i^{(k)} (i = 1,...,n)$. If also (iii) $0 \notin \underline{d}_i$ where $\underline{d}_i \in I(R)$ is such that $p'(x) \in \underline{d}_i\left(\forall x \in x_i^{(0)}\right)\left(i = 1,...,n\right)$, then $x_i^{(k)} \to x_i^*\left(k \to \infty\right)\left(i = 1,...,n\right)$ and (13) holds. Furthermore, for $i = 1,...,n$, $O_R\left(IS1, x_i^*\right) \geq 1 + \sigma$ where $\sigma \in (1,2)$ is the greatest positive zero of $t^n - t - 1$. □

THE INTERVAL SYMMETRIC SINGLE-STEP *ISS1*

A natural extension of the interval single-step procedure *IS1* is the interval symmetric single-step procedure *ISS1* which is based on the symmetric single-step idea Aitken (1950) and Alefeld (1977), and may be regarded as an interval version of the point procedure *PSS1* in Monsi (2010). The procedure *ISS1* consists of generating the sequences $\left\{\underline{x}_i^{(k)}\right\}\left(i = 1,...,n\right)$ from

$$\underline{x}_i^{(k,0)} = \underline{x}_i^{(k)} \quad \left(i = 1,...,n\right), \qquad (15a)$$

$$x_i^{(k,0)} = m\left(\underline{x}_i^{(k)}\right) \quad \left(i = 1,...,n\right), \qquad (15b)$$

$$p_i^{(k)} = p\left(x_i^{(k)}\right) \quad \left(i = 1,...,n\right), \qquad (15c)$$

$$\underline{x}_i^{(k)} = \left\{ x_i^{(k)} - \frac{p_i^{(k)}}{\Pi_{j=1}^{i-1}\left(x_i^{(k)} - \underline{x}_j^{(k,1)}\right)\Pi_{j=i+1}^{n}\left(x_i^{(k)} - \underline{x}_j^{(k,0)}\right)} \right\} \cap \underline{x}_i^{(k,0)}, \qquad \text{(15d)}$$

$$\left(i = 1,\ldots,n\right),$$

$$\underline{x}_i^{(k,2)} = \left\{ x_i^{(k)} - \frac{p_i^{(k)}}{\Pi_{j=1}^{i-1}\left(x_i^{(k)} - \underline{x}_j^{(k,1)}\right)\Pi_{j=i+1}^{n}\left(x_i^{(k)} - \underline{x}_j^{(k,2)}\right)} \right\} \cap \underline{x}_i^{(k,1)}, \qquad \text{(15e)}$$

$$\left(i = n,\ldots,1\right),$$

$$\underline{x}_i^{(k+1)} = \underline{x}_i^{(k,2)} \left(i = 1,\ldots,n\right)\left(k \geq 0\right). \qquad \text{(15f)}$$

The procedure *ISS1* has the following attractive features:

- The values $p\left(x_i^{(k)}\right)\left(i = 1,\ldots,n\right)$ which are computed for use in (15d) are re-used in (15e).
- The products $\Pi_{j=1}^{i-1}\left(x_i^{(k)} - \underline{x}_j^{(k,1)}\right)\left(i = 2,\ldots,n\right)$ which are computed for use in (15d) are re-used in (15e).
- $\underline{x}_n^{(k,1)} = \underline{x}_n^{(k,2)}\left(k \geq 0\right)$ so that $\underline{x}_n^{(k,2)}$ need not be computed.
- The R-order of convergence of the interval total-step *IT1* procedure defined by (12) is at least 2 or $O_R\left(IT1\right) \geq 2$.

The interval single-step *IS1* procedure (steps (14a)-(14b)) has been studied by Alefeld and Herzberger (1983). The R-order of convergence $O_R\left(IS1, x^*\right)$ for *IS1* to the set of simple zeros $x^* = \left(x_1^*, x_2^*, \ldots, x_n^*\right)^T$ is such that $O_R\left(IS1, x^*\right) \geq 1 + \tau > 2$, where $\tau \in \left(1,2\right)$ is the unique positive zero of $t^n - t - 1$. As shown subsequently in this paper that the corresponding R-order of convergence of *ISS1* defined by (15) is at least 3 or $O_R\left(ISS1, x^*\right) \geq 3$.

Theorem 4

If (i) (2) and (3) hold; (ii) the sequences $\left\{\underline{x}_i^{(k)}\right\}(i=1,\ldots,n)$ are generated from (15), then $(\forall k \geq 0)\, x_i^* \in \underline{x}_i^{(k+1)} \subseteq \underline{x}_i^{(k)} = (i=1,\ldots,n)$.

If also (iii) $0 \notin \underline{d}_i \in I(R)$ is such that $p'(x) \in \underline{d}_i \left(\forall x \in \underline{x}_i^{(0)}\right) = (i=1,\ldots,n)$, then $\underline{x}_i^{(k)} \to x_i^* \,(k \to \infty)(i=1,\ldots,n)$ and (13) holds. Then for $(i=1,\ldots,n), O_R\left(ISS1, x_i^*\right) \geq 3$.

Proof

The proof that $x_i^* \in \underline{x}_i^{(k+1)} \subseteq \underline{x}_i^{(k)}\,(i=1,\ldots,n)(\forall k \geq 0)$ and that (13) holds is almost identical with the corresponding proofs in Theorem 1 and Theorem 2, and is therefore omitted. It remains to prove that for $(i=1,\ldots,n), O_R\,(ISS1), x_i^* \geq 3$.

As in the proof of Theorem 2 (Alefeld and Herzberger (1983)) it may be shown that $\exists \alpha > 0$ such that $(\forall k \geq 0)$,

$$w_i^{(k,1)} \leq \beta w_i^{(k,0)} \left\{ \sum_{j=1}^{i-1} w_j^{(k,1)} + \sum_{j=i+1}^{n} w_j^{(k,0)} \right\} \; (i=1,\ldots,n), \tag{16}$$

and

$$w_i^{(k,2)} \leq \beta w_i^{(k,0)} \left\{ \sum_{j=1}^{i-1} w_j^{(k,1)} + \sum_{j=i+1}^{n} w_j^{(k,2)} \right\} \; (i=n,\ldots,1), \tag{17}$$

where

$$w_i^{(k,s)} = (n-1)\alpha w\left(\underline{x}_i^{(k,s)}\right)(s=0,1,2), \tag{18}$$

and

$$\beta = \frac{1}{n-1}. \tag{19}$$

Let

$$u_i^{(1,1)} = \begin{cases} 2 & (i = 1,\dots,n-1) \\ 3 & (i = n) \end{cases}, \tag{20}$$

and

$$u_i^{(1,2)} = \begin{cases} 4 & (i = 1) \\ 3 & (i = 2,\dots,n) \end{cases}, \tag{21}$$

and for $r = 1,2$ let

$$u_i^{(k+1,r)} = \begin{cases} 3u_i^{(k,r)} + 1 & (i = 1) \\ 3u_i^{(k,r)} & (i = 2,\dots,n) \end{cases}. \tag{22}$$

Then by (20) – (22), for $(\forall k \geq 0)$,

$$u_i^{(k,1)} = \begin{cases} \frac{5}{2}\left(3^{k-1}\right) - \frac{1}{2} & (i = 1) \\ 2\left(3^{k-1}\right) & (i = 2,\dots,n-1), \\ 3\left(3^{k-1}\right) & (i = n) \end{cases} \tag{23}$$

and

$$u_i^{(k,2)} = \begin{cases} \frac{9}{2}\left(3^{k-1}\right) - \frac{1}{2} & (i = 1) \\ 3\left(3^{k-1}\right) & (i = 2,\dots,n) \end{cases}. \tag{24}$$

Suppose, without loss of generality, that

$$w_i^{(0,0)} \le h < 1 \quad \left(i = 1, \ldots, n\right). \tag{25}$$

Then by an inductive argument it follows from (16) – (25) that for $\left(i = 1, \ldots, n\right)\left(k \ge 0\right)$,

$$w_i^{(k,1)} \le h^{u_i^{(k+1,1)}},$$

and

$$w_i^{(k,2)} \le h^{u_i^{(k+1,2)}},$$

whence, by (24) and (15f), $\left(\forall k \ge 0\right)$

$$w_i^{(k+1)} \le h^{3^{(k+1)}} \quad \left(i = 1, \ldots, n\right)$$

So $\left(\forall k \ge 0\right)$, by (17) – (25),

$$w\left(\underline{x}_i^{(k)}\right) \le \left(\frac{\beta}{\alpha}\right) h^{3^k} \quad \left(i = 1, \ldots, n\right). \tag{26}$$

Let

$$w^{(k)} = \max_{1 \le i \le n} \left\{ w\left(\underline{x}_i^{(k)}\right) \right\}.$$

Then by (26),

$$w^{(k)} \le \left(\frac{\beta}{\alpha}\right) h^{3^k} \quad \left(\forall k \ge 0\right).$$

So

$$\begin{aligned} R_3\left(w^{(k)}\right) &= \lim_{k\to\infty} \sup\left\{\left(w^{(k)}\right)^{1/(3^k)}\right\} \\ &= \lim_{k\to\infty}\left\{\left(\frac{\beta}{\alpha}\right)^{1/(3^k)h}\right\} \\ &= h \\ &< 1. \end{aligned}$$

Therefore , it follows from Alefeld and Herzberger (1983), Orthega and Rheindfold (1970) that

$$O_R\left(ISS1, x_i^*\right) \ge 3 \quad \left(i = 1,\ldots,n\right). \ \square$$

NUMERICAL RESULTS

The following examples are used to compare the efficiencies of the procedures *IT1, IS1* and *ISS1.*

Example 1:

The characteristic polynomial

$$p(\lambda) = \det(\lambda I - A), \tag{27a}$$

where

$$A = \begin{pmatrix} a_1 & b_1 & & 0 & \\ b_1 & a_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & a_{n-1} & b_{n-1} \\ & & & b_{n-1} & a_n \end{pmatrix}$$

and

$$f^{(0)}(\lambda)=1$$
$$f^{(1)}(\lambda)=(\lambda-a_1), \tag{27b}$$
$$f^{(k)}(\lambda)=(\lambda-a_k)f^{(k-1)}(\lambda)-(b_{k-1})^2 f^{(k-2)}(\lambda)(2\le k\le n),$$

$$p(\lambda)=f^{(n)}(\lambda).$$

For this example (Alefeld and Herzberger (1983)):

$$n=9,$$
$$b_i=1 \quad (i=1,\ldots,n-1),$$
$$a_1=15;\ a_2=10;\ a_3=7;\ a_4=4,$$
$$a_5=0;\ a_6=-4;\ a_7=-7;\ a_8=-10;\ a_9=-15.$$

Initial intervals:

$$\underline{x}_1^{(0)}=[14,16],\ \underline{x}_2^{(0)}=[8,12],\ \underline{x}_3^{(0)}=[5,9],$$
$$\underline{x}_4^{(0)}=[2,6],\ \underline{x}_5^{(0)}=[-2,2],\ \underline{x}_6^{(0)}=[-6,-2],$$
$$\underline{x}_7^{(0)}=[-9,-5],\ \underline{x}_8^{(0)}=[-12,-8],\ \underline{x}_9^{(0)}=[-17,-12].$$

<u>Example 2</u> (Alefeld and Herzberger (1983))

The polynomial is given by (27) with

$$n=5,$$
$$a_1=12,\ a_2=9,\ a_3=6,\ a_4=3,\ a_5=0,$$
$$b_i=1 \quad (i=1,\ldots,4).$$

Initial intervals:

$$\underline{x}_1^{(0)}=[11,13],\ \underline{x}_2^{(0)}=[7,11],\ \underline{x}_3^{(0)}=[4,8],$$
$$\underline{x}_4^{(0)}=[1,5],\ \underline{x}_5^{(0)}=[-1,1].$$

Example 3

The polynomial is given by (27) with

$$n = 9,$$
$$a_1 = 10 \ \ (i = 1,...,9),$$
$$b_i = 20 \ \ (i = 1,...,8),$$

The zeros: $x_i^* = 10 + 40 \ \cos \ \left(\frac{i\pi}{n+1}\right) \ (i = 1,...,n).$

Initial intervals: $\underline{x}_i^{(0)} = \left[x_i^* - 2.8, x_i^* + 5.6\right] \ (i = 1,...,n).$

Example 4

The polynomial is as in Example 3 save that in this example,

$$a_1 = -10 \ \ (i = 1,...,n).$$

Example 5

The polynomial is as equation (4).

The zeros:

$$x_i^* = \begin{cases} -2\left(\frac{n}{2} - i + 1\right) & (i = 1,..., \frac{n}{2}), \\ -x_{n-i+1}^* & (i = \frac{n}{2} + 1,...,n). \end{cases}$$

Initial intervals: $\underline{x}_i^{(0)} = \left[x_i^* - 0.5, x_i^* + 1.0\right] \ (i = 1,...,n).$

TABLE 1: CPU times in seconds.

Example	***n***	***IT1***	***IS1***	***ISS1***
1	9	3.67	3.06	2.92
2	5	1.23	1.15	1.14
3	9	4.28	3.80	3.65
4	9	4.41	3.71	3.71
5	14	9.76	8.09	6.27

TABLE 2: Number of iterations.

Example	***n***	***IT1***	***IS1***	***ISS1***
1	9	5	4	3
2	5	4	4	3
3	9	6	5	4
4	9	6	5	4
5	14	6	5	3

CONCLUSION

We have shown analytically that the interval symmetric single-step procedure *ISS1* gives better results in terms of the rate of convergence, where the *R*-order of convergence of *ISS1* is at least 3 or $O_R\left(ISS1, x^*\right) \geq 3.$

On the other hand, the *R*-order of convergence of *IS1* of Alefeld and Herzberger (1983) is greater than 2, that is $O_R\left(IS1, x^*\right) > 2,$ and also that the *R*-order of convergence of *IT1* of Kerner (1966) is at least 2 or $O_R\left(IT1, x^*\right) \geq 2.$

It is clear from Table 1 and Table 2 that the procedure *ISS1* numerically requires less CPU times and number of iterations then does *IT1* and *IS1*. These procedures have been implemented in Triplex S-algol (Cole and Morrison (1982)) on a VAX 11-785 computer. The stopping criterion used is $w^{(k)} \leq 10^{-10}.$

REFERENCES

Aberth, O. 1973. Iteration methods for finding all zeros of a polynomial simultaneously. *Maths. of Comput*. **27**: 339-344.

Aitken, A.C. 1950. Studies in practical mathematics V. On the iterative solution of linear equation. *Proc. Roy. Soc. Edinburg Sec. A*. **63**: 52-60.

Alefeld, G. 1977. The symmetric single-step method for systems of simultaneous linear equations with intervals as coefficients. *Computing.* **18**: 329-340.

Alefeld, G. and Herzberger, J. 1983. *Introduction to Interval Computations,* New York: Academic Press.

Cole, A.J. and Morrison, R. 1982. Triplex: A system for interval arithmetic. *Software – Practice and Experience.* **12**: 341-350.

Gargantini, I. 1975. Parallel square root iterations, Interval Mathematics K. Nickel, (Ed.). *Lecture Notes in Computer Sciences 29*. Heidelberg: Springer Verlag,

Gargantini, I. 1976. Parallel Laguerre iterations: The complex case. *Numer. Math.* 26: 317-323.

Gargantini, I. 1978. Further applications of circular arithmetic: Schroeder-like algorithms with error bound for finding zeros of polynomials. *SIAM J. Numer. Anal.* **15**: 497-510.

Gargantini, I. 1981. *An application of interval mathematics: A polynomial solver with degree four convergence,* Freiburger Intervallbericht 81/7.

Garganti, I. and Henrici, P.1972. Circular arithmetic and the determination of polynomial zeros. *Numer. Math.***18**: 305-320.

Glatz, G. 1975. Newton algorithms for the determination of polynomial roots using complex circular arithmetic, Interval Mathematic K. Nickel,(Ed.). *Lecture Notes in Computer Science 29*. Heidelberg: Springer Verlag.

Henrici, P. 1974. *Applied and Computational Complex Analysis*. New York: John Wiley and Sons.

Kerner, O. 1966. Total step procedure for the calculation of the zeros of polynomials. *Numer. Math.* **8**: 290-294.

Krier, N. and Spellucci, P. 1975. Inclusion sets of polynomial zeros. Interval Mathematics K. Nickel, (Ed.). *Lecture Notes in Computer Science 29*. Heidelberg: Springer Verlag.

Milovanovic, G.V. and Petkovic, M.S. 1983. On the convergence of a modified method for simultaneous finding of polynomial zeros. *Computing*. **30**: 171-178.

Monsi, M. 2010. The Point Symmetric Single-Step PSS1 Procedure for Simultaneously Estimating Simple Polynomial Zeros. (Submitted to the Malaysian Journal of Mathematical Sciences).

Moore, R.E.1962. *Interval Arithmetic and Automatic Error Analysis in Digital Computing,* PhD Thesis, Stanford University.

Moore, R.E.1979. *Methods and Applications of Interval Analysis.* Philadelphia: SIAM Publications.

Ortega, J.M. and Rheinboldt,W.C. 1970. *Iterative Solution of Nonlinear Equations in Several Variables,* New York: Academic Press.

Petkovic, M.S. 1980. On the generalization of some algorithms for the simultaneous approximation of polynomial roots in interval mathematics K. Nickel, (Ed.). New York: Academic Press.

Petcovic, M.S. 1982. On an iterative method for simultaneous inclusion of polynomial complex zeros. *J. Computational and Appl. Math.* **8**: 51-52.

Petkovic, M.S. and Milovanovic,G.V. 1983. A note on some improvements of the simultaneous methods for determination of polynomial zeros. *J. Computational and Applied. Math.* **9**: 65-69.

Petkovic, M.S. and Stefanovic, L.V. 1986. On a second order method for the simultaneous inclusion of polynomial complex zeros. *J. Comp. and Appl. Math.***15**: 13-25.

Petkovic, M.S. and Stefanovic, L.V.1986. On some improvements of square root iteration for polynomial complex zeros. *J. Comp. and Appl. Math*. **15**:13-25.

Petkovic, M.S. and Stefanovic, L.V. 1987. On some iteration functions for the simultaneous computation of multiple complex polynomial zeros. *BIT*. **27**: 111-122.